Exponential Recursion for Multi-Scale Problems in Electromagnetics

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Computational Aspects of Time Dependent Electromagnetic Wave Problems in Complex Materials
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Outline

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Exponential recursion is a simple concept, but a powerful tool.
We focus on EM wave propagation in complex media.

(1) Anomalous dielectric relaxation (fractional relaxation models)

1. Complex hetergoeneous materials (soil, biological tissues)
2. Empirical dispersion models involve $(i\omega)^{\alpha}$.
3. Power-law decay, requires time history of fields.
4. Useful for similar problems in acoustics, solid mechanics, etc.

(2) Plasmas

1. Plasma phenomena occur at vastly disparate time scales.
2. Geometry, fine spatial scales must be resolved.
3. Experiments often have non-local effects.
4. Pros and Cons of kinetic vs. fluid models of plasma.

In time, exponential recursion truncates time history. Consider

$$\dot{\phi} + y\phi = f(t), \qquad 0 < t < T$$

Multiply by integrating factor, and integrate over $[t - \delta, t]$

$$\int_{t-\delta}^{t} \left(e^{yt}\phi\right)' \, dt = \int_{t-\delta}^{t} e^{yt} f(t) \, dt$$

$$e^{yt}\phi(t) - e^{y(t-\delta)}\phi(t-\delta) = \int_{t-\delta}^{t} e^{y\tau} f(\tau) \, d\tau$$

Rearranging, we find the exponential recursion

$$\phi(t) = e^{-y\delta}\phi(t-\delta) + \int_{0}^{\delta} e^{-yu} f(t-u) \, du.$$

Discretize using any exponential integrator.

In space, exponential recursion localizes the solution. Consider

$$w - \frac{1}{\alpha^2} w'' = u \qquad \Longrightarrow \qquad w_p(x) = \frac{\alpha}{2} \int_a^b e^{-\alpha|x-y|} u(y)\, dy$$

Split the integral at $y = x$, and let $w_p = w_L + w_R$. Then

$$w_L(x) = \frac{\alpha}{2} \int_a^x e^{-\alpha(y-x)} u(y)\, dy, \qquad w_R(x) = \frac{\alpha}{2} \int_x^b e^{-\alpha(x-y)} u(y)\, dy.$$

A few steps of algebra produce the exponential recursion

$$w_L(x) = e^{-\alpha\delta_L} w_L(x - \delta_L) + \int_0^{\delta_L} e^{-\alpha z} u(x - z)\, dz$$

$$w_R(x) = e^{-\alpha\delta_R} w_R(x + \delta_R) + \int_0^{\delta_R} e^{-\alpha z} u(x + z)\, dz.$$

Discretize using any collocation method.

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Consider EM propagation through a region Ω

$$\frac{\partial D}{\partial t} = \nabla \times H - J, \qquad \nabla \cdot D = \rho$$
$$\frac{\partial B}{\partial t} = -\nabla \times E, \qquad \nabla \cdot B = 0$$

- In the absence of external charges and currents, $\rho = 0$, $J = \mathbf{0}$.
- Assume the material is non-magnetic, so $B = \mu_0 H$.
- Anomalous dielectric relaxation, $\hat{D} = \epsilon_0 \hat{\epsilon} \hat{E}$, where

$$\hat{\epsilon}(s) = \epsilon_\infty + \frac{\Delta\epsilon}{(1 + (s\tau)^\alpha)^\beta}$$

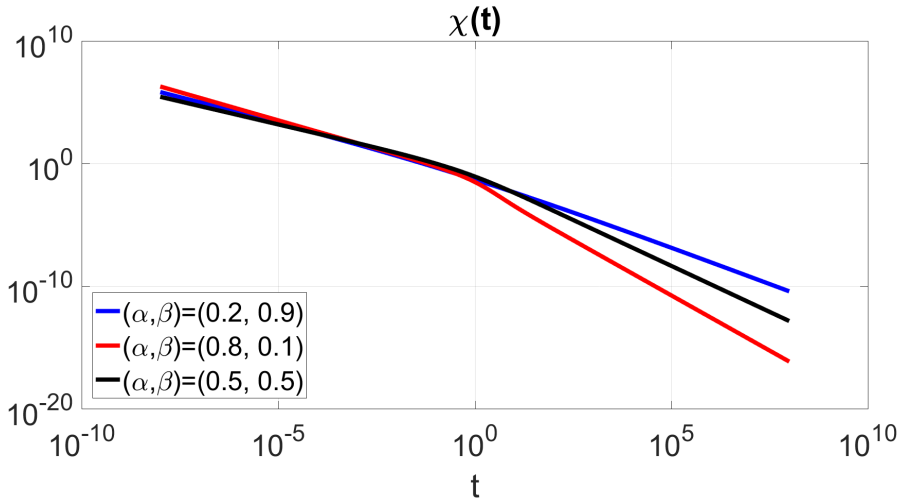
with $\epsilon_\infty \geq 1, \quad \Delta\epsilon > 0, \quad 0 < \alpha, \beta \leq 1.$

Decompose $D = \epsilon_0 \left(\epsilon_\infty E + P\right)$, where

$$\left(I + \tau^\alpha \, {}_c D_t^\alpha\right)^\beta P = \Delta\epsilon E \qquad \textit{fractional PDO!}$$

Solve for the polarization using Laplace transforms.

$$P(x,t) = \int_0^t \chi(u) E(x, t-u)\, du, \qquad \hat{\chi}(s) = \frac{\Delta\epsilon}{\left(1 + (s\tau)^\alpha\right)^\beta}$$



$$\chi(t) \sim \begin{cases} t^{\alpha\beta - 1} & t \ll 1 \\ t^{-\alpha - 1} & t \gg 1 \end{cases}$$

We first re-cast the susceptibility as an integral over $\mathbb{R}$

$$\begin{aligned}\chi(t) =& \frac{\Delta\epsilon}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\frac{e^{st}}{(1+(s\tau)^{\alpha})^{\beta}}ds \\ =& \int_0^{\infty} f(y)e^{-yt/\tau}\,dy \\ =& \int_{-\infty}^{\infty} f(e^z)e^{z-e^z t/\tau}\,dz,\end{aligned}$$

where

$$f(y) = \frac{\Delta\epsilon}{\pi\tau}\frac{\sin\left(\beta\cos^{-1}\left\{\frac{y^{\alpha}\cos(\pi\alpha)+1}{\sqrt{y^{2\alpha}+2\cos(\pi\alpha)y^{\alpha}+1}}\right\}\right)}{(y^{2\alpha}+2\cos(\pi\alpha)y^{\alpha}+1)^{\beta/2}}.$$

In general, consider

$$K(t) = \int_0^\infty f(y)e^{-yt}\,dy$$

for non-negative $f \in L^1(\mathbb{R}^+)$. We set $y = e^z$. Then, for $h > 0$ the Poisson summation formula yields

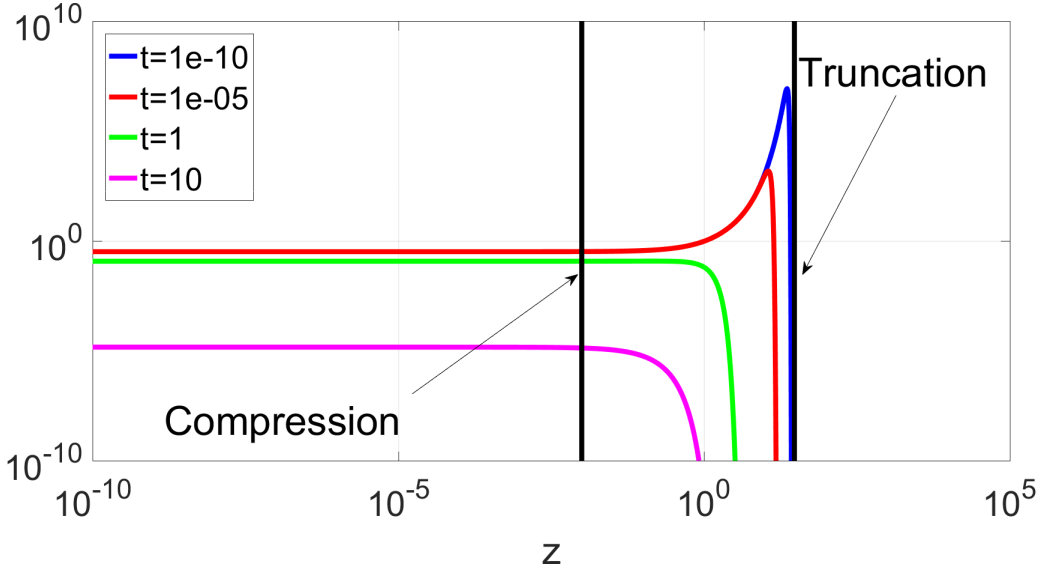
$$h \sum_{n=-\infty}^{\infty} f(e^{nh})e^{nh-e^{nh}t} = \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi k}{h}\right)$$

where

$$\hat{f}(k) = \int_{-\infty}^{\infty} \left[f(e^z)e^{z-e^z t}\right] e^{ikz}\,dz, \qquad \hat{f}(0) = K(t)$$

Under mild assumptions, the integrand is analytic for $Im\{z\} \leq \theta$.

$$|\hat{f}(k)| \leq C_k e^{-|k|\theta}.$$



$$K(t) = h \sum_{n=-\infty}^{\infty} f(e^{nh}) e^{nh - e^{nh} t} - \sum_{k \neq 0} \hat{f}\left(\frac{2\pi k}{h}\right)$$

$$\approx h \sum_{nh < z_r} f(e^{nh}) e^{nh - e^{nh} t} + \mathcal{O}\left(e^{-\pi^2/h}\right)$$

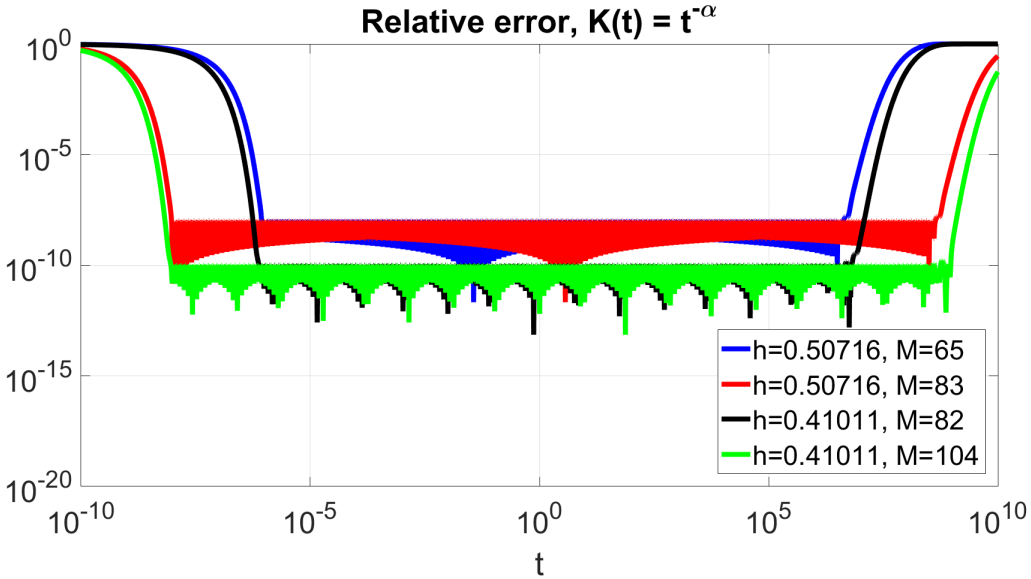
Given $\epsilon > 0$, for $t \in [\Delta t, T]$

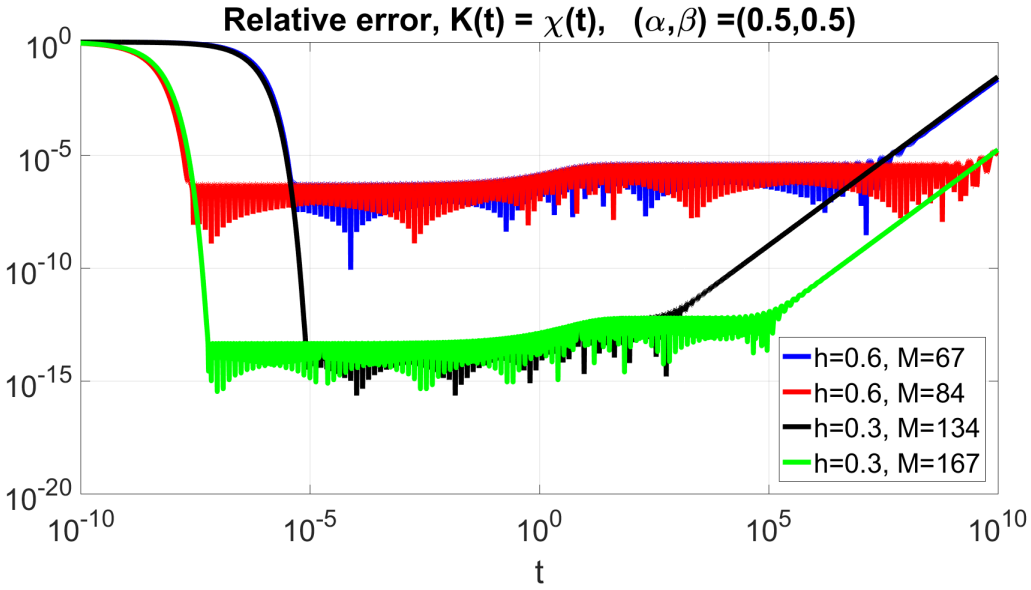
1. **Discretization:** Choose h so that $e^{-\pi^2/h} \approx \epsilon$.
2. **Truncation:** Choose z_r so that $hf(e^{z_r})e^{z_r - \Delta t e^{z_r}} \approx \epsilon$.
3. **Compression:** Choose z_ℓ so that a minimal number J (typically 2 or 3) compressed nodes satisfies

$$h \sum_{z \le z_\ell} f(e^{nh}) e^{nh - e^{nh} T} = \sum_{j=1}^{J} w_j e^{-y_j T} + \mathcal{O}(\epsilon)$$

Next, merge the weights and nodes. Then, we have a uniform relative error bound

$$\max_{\Delta t < t < T} \left| K(t) - \sum_{m=1}^{M} w_m e^{-y_m t} \right| \le K(t)\epsilon$$


Relative error, K(t) = t^-α
10^0
10^-5
10^-10
10^-15
10^-20
10^-10
10^-5
10^0
10^5
10^10
t
h=0.50716, M=65
h=0.50716, M=83
h=0.41011, M=82
h=0.41011, M=104


Relative error, K(t) = χ(t), (α,β) =(0.5,0.5)
10^0
10^-5
10^-10
10^-15
10^-20
10^-10
10^-5
10^0
10^5
10^10
t
h=0.6, M=67
h=0.6, M=84
h=0.3, M=134
h=0.3, M=167

For $t \in [0, T]$,

$$P(x,t) = \int_0^{\Delta t} \chi(u) E(x, t-u)\, du + \sum_{m=1}^{M} w_m \phi_m(t) + \mathcal{E}(t)$$

where we have the diagonalized linear system

$$\dot{\phi}_m + y_m \phi_m = E.$$

Discretize using any exponential integrator. For $E \in L^2([0,T])$,

$$\begin{aligned} ||\mathcal{E}(t)||_{L^2} &\leq \left\| \chi(t) - \sum_{m=1}^{M} w_m e^{-y_m t} \right\|_{L^1} ||E(x,t)||_{L^2} \\ &\leq T \left\| \chi(t) - \sum_{m=1}^{M} w_m e^{-y_m t} \right\|_{L^\infty} ||E(x,t)||_{L^2} \\ &\leq \epsilon T ||E(x,t)||_{L^2} \end{aligned}$$

Replace the electric field with a polynomial interpolant, and perform product integration to arrive at[1]

$$P(t) = \sum_{\ell=0}^{L} A_\ell E(x, t - \ell\Delta t) + \sum_{m=1}^{M} w_m \phi_m(t)$$

$$\phi_m(x,t) = e^{-y_m \Delta t}\phi_m(x, t - \Delta t) + \sum_{\ell=0}^{L} B_{\ell,m} E(x, t - \ell\Delta t).$$

The polarization law can now be evaluated with L levels of time history, and $M = \mathcal{O}(\log N_t)$ terms in memory.

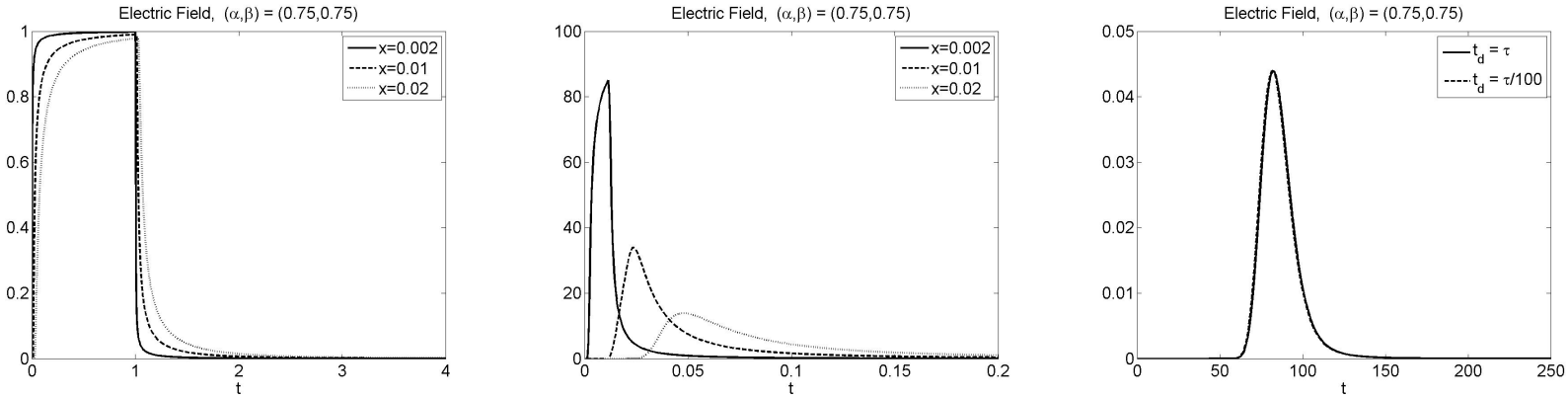
The operation count is $\mathcal{O}(LM)$. Here, $N_t = \lceil \frac{T}{\Delta t} \rceil$.

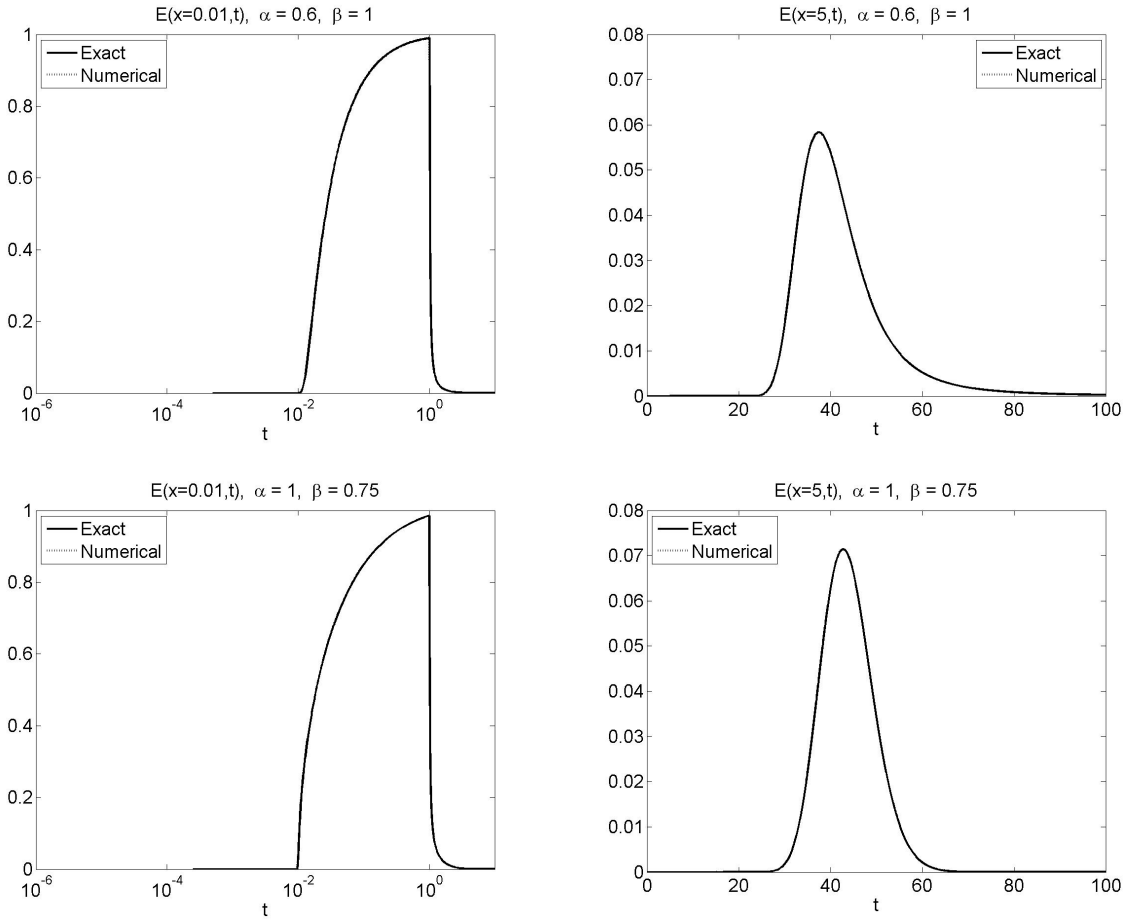
[1] JCP 2013, with S. Jiang and P. Petropoulos

Theorem. The numerical scheme, based on the second order accurate FDTD method, and is stable under the standard CFL stability condition.

We solve a signaling problem (1d TEM wave) using the finite difference time domain (FDTD) technique, with square pulse

$$E(0,t) = \frac{1}{t_d}\left(H(t) - H(t - t_d)\right).$$





Similar asymptotics for Cole-Cole, Cole-Davidson models. [2]

[2] IEEE Trans. Ant. 2011 with P. Petropoulos

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Consider the Vlasov-Maxwell system for a plasma

$$\epsilon_0 \mu_0 \frac{\partial E}{\partial t} = \nabla \times B - \mu_0 J, \qquad \nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \qquad \nabla \cdot B = 0$$

$$\rho = q \int_v f(x, v, t)\, dv, \qquad J = q \int_v v f(x, v, t)\, dv$$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_v f = 0$$

$$F = q\,(E + v \times B)$$

Define EM fields using scalar potential ϕ and vector potential A by

$$E = -\frac{\partial A}{\partial t} - \nabla\phi, \quad B = \nabla \times A.$$

Impose the Lorenz gauge $\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0$, so that

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J$$
$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{1}{\epsilon_0}\rho$$

1. **Apply the method of lines transpose (MOL^T)**
   - Discretize in time first.
   - Re-formulate as a semi-discrete boundary integral.

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   - Spatial solver works in a "line-by-line" fashion.
   - Embedded boundaries.
   - Local interpolation for normal derivatives.

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   - Embedded boundaries.
   - Local interpolation for normal derivatives.
4. **Raise order of accuracy in time**
   - Taylor expansion reformulated with convolution operators.
   - Coefficients determined with resolvent expansion.
   - Stability guaranteed by introducing a free parameter $\beta > 0$.

$$\frac{1}{c^2} u_{tt} - \nabla^2 u = S$$

[2] Math Comp. 2014 with A. Christlieb, B. Ong, and **L. Van Groningen**

$$\frac{1}{c^2} u_{tt} - \nabla^2 u = S$$

Discretize u_{tt} in time

$$u_{tt}(x, t_n) \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}.$$

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$$\frac{1}{c^2}u_{tt} - \nabla^2 u = S$$

Discretize u_{tt} in time

$$u_{tt}(x, t_n) \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}.$$

The Laplacian is treated semi-implicitly, so that

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(c\Delta t^2)} - \nabla^2 \left(u^n + \frac{u^{n+1} - 2u^n + u^{n-1}}{\beta^2} \right) \approx S^n,$$

with $0 < \beta \leq 2$.

[2] Math Comp. 2014 with A. Christlieb, B. Ong, and **L. Van Groningen**

$$\mathcal{L}\left[u^n + \frac{u^{n+1} - 2u^n + u^{n-1}}{\beta^2}\right] = u^n + \frac{S^n}{\alpha^2},$$

where the modified Helmholtz operator is

$$\mathcal{L}[u](x) := \left(1 - \frac{1}{\alpha^2}\nabla^2\right) u(x), \quad \alpha = \frac{\beta}{c\Delta t}.$$

Update equation

$$u^{n+1} = 2u^n - u^{n-1} - \beta^2 D[u^n](x) + \beta^2 \mathcal{L}^{-1}\left[\frac{S^n}{\alpha^2}\right](x),$$

where

$$D[u](x) := u(x) - \mathcal{L}^{-1}[u](x) \approx -\frac{1}{\alpha^2}\nabla^2 u.$$

Rather than invert the 3d Helmholtz operator, we utilize dimensional splitting

$$\mathcal{L} \approx \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z$$

In one spatial dimension,

$$u^{n+1} = 2u^n - u^{n-1} - \beta^2 D[u^n](x) + \beta^2 \mathcal{L}^{-1}\left[\frac{S^n}{\alpha^2}\right](x),$$

where

$$\mathcal{L}^{-1}[u](x) := \underbrace{\frac{\alpha}{2}\int_a^b u(y)e^{-\alpha|x-y|}dy}_{\text{Particular Solution}} + \underbrace{Ae^{-\alpha(x-a)} + Be^{-\alpha(b-x)}}_{\text{Homogeneous Solution}}.$$

Theorem: This semi-discrete scheme is second order accurate in time, and A-stable for $\beta \in (0, 2]$.

$$w_p(x) = I[u](x) = \frac{\alpha}{2}\int_a^b e^{-\alpha|x-y|}u(y)dy.$$

Split the particular solution at $y = x$,

$$I[u](x) = \underbrace{\frac{\alpha}{2}\int_a^x e^{-\alpha(x-y)}u(y)dy}_{I^L(x)} + \underbrace{\frac{\alpha}{2}\int_x^b e^{-\alpha(y-x)}u(y)dy}_{I^R(x)}.$$

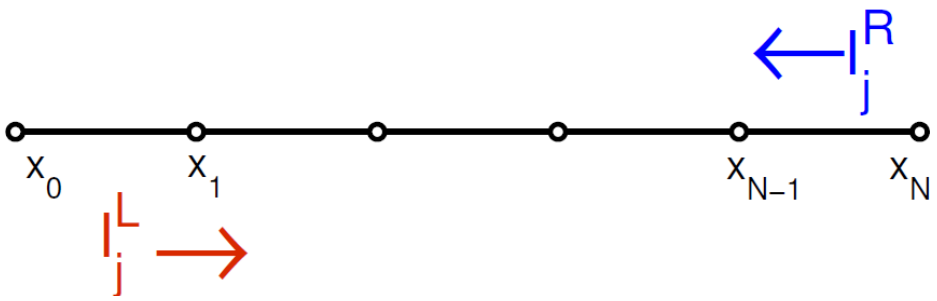
Each "characteristic" is updated using **exponential recursion**

$$\begin{aligned} I^L(x+\delta) =& e^{-\alpha\delta}I^L(x) + \frac{\alpha}{2}\int_x^{x+\delta} e^{-\alpha(x_j-y)}u(y)dy \\ I^R(x-\delta) =& e^{-\alpha\delta}I^R(x) + \frac{\alpha}{2}\int_{x-\delta}^{x} e^{-\alpha(y-x_j)}u(y)dy. \end{aligned}$$

Partition $[a, b]$ into N subintervals $[x_{j-1}, x_j]$, $h_j = x_j - x_{j-1}$.
Replace u with a local Lagrange interpolant, of order $2M$. Then

$$I^L(x_j) = e^{-\alpha h_j} I^L(x_{j-1}) + \sum_{k=-M}^{M} w_k^L u(x_{j+k}), \qquad j = 1, \ldots N,$$

$$I^R(x_j) = e^{-\alpha h_{j+1}} I^R(x_{j+1}) + \sum_{k=-M}^{M} w_k^R u(x_{j+k}), \quad j = N-1, \ldots 0.$$



Convolution computed in $O(MN)$ operations.

Transmission conditions can be formulated, for domain decomposition and (1d) outflow BCs.

$$\mathcal{L}^{-1}[u](x) := \underbrace{\frac{\alpha}{2}\int_a^b u(y)e^{-\alpha|x-y|}dy}_{\text{Particular Solution}} + \underbrace{Ae^{-\alpha(x-a)} + Be^{-\alpha(b-x)}}_{\text{Homogeneous Solution}}.$$

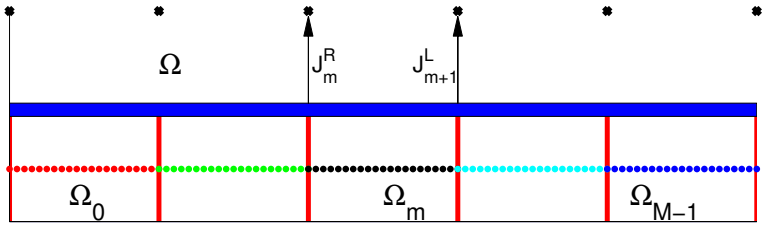
Compare with the free space solution. Then, for $x \in [a, b]$, we see

$$A(t) = \alpha \int_0^\infty u(a-y,t)e^{-\alpha y}dy.$$

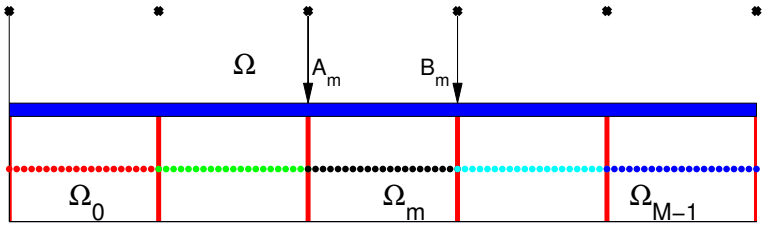
Using **exponential recursion**, storing time history at the boundary

$$\begin{aligned} A^n &= \alpha \int_{a-ct_n}^{a} e^{-\alpha(a-y)}u(y,t_n)dy \\ &= e^{-\beta}A^{n-1} + \alpha \int_0^{c\Delta t} e^{-\alpha y} u\left(a, t_n - \frac{y}{c}\right) dy. \end{aligned}$$

The exponential recursion can also be employed hierarchically.[3]



(a) Fine-Coarse Pass



(b) Coarse-Fine Pass

$$I^L(X_j) = e^{-\alpha(X_j - X_{j-1})} I^L(X_{j-1}) + J^L(X_j),$$

$$J^L(X_j) = \alpha \int_{X_{j-1}}^{X_j} e^{-\alpha(X_j - y)} u(y) dy$$

1. Compute each local particular solution on Ω_j.
2. Find particular solution using global exponential recursion.
3. Pass transmission conditions, boundary conditions to each Ω_j.

[3] Arxiv, with A. Christlieb, **Y. Guclu** and E. Wolf

CPU vs GPU Run Time Data

dx	CFL	# Grid Points	CPU Run Time (Sec)	GPU 1 Domain Run Time (Sec)	GPU 4 domain Run Time (Sec)	GPU 8 domain Run Time (Sec)	GPU 16 domain Run Time (Sec)
0.125	0.08	32	0.45	5.08	5.07	5.17	5.52
0.03125	0.32	128	1.74	5.14	5.08	5.18	5.3
0.015625	0.64	256	3.45	5.26	5.168	5.21	5.35
0.0078125	1.28	512	6.91	5.87	5.52	5.52	5.64
0.00390625	2.56	1024	27.8	7.08	5.68	5.65	5.85
0.001953125	5.12	2048	53	hardware limitation	6.076	5.88	5.89
0.0009765625	10.24	4096	104	hardware limitation	8.01 (unstable)	6.76 (unstable)	6.38(unstable)

2D CPU vs GPU runtime comparison

Grid Width	CPU Run Time (Sec)	GPU Run Time (Sec)	GPU Speed Up %
32	1.57	1.3	120.7692308
64	6	1.25	480
128	25	1.88	1329.787234
256	93	3	3100

Undergraduate thesis with C. Seipp

In order to approximate higher order powers of the Laplacian operator using dimensional splitting[4], we first define

$$\mathcal{L}_\gamma := 1 - \frac{\partial_\gamma^2}{\alpha^2}, \quad \mathcal{D}_\gamma := 1 - \mathcal{L}_\gamma^{-1}, \quad \gamma = x, y, z,$$

and

$$\mathcal{C}_{xyz} := \mathcal{L}_y^{-1}\mathcal{L}_z^{-1}\mathcal{D}_x + \mathcal{L}_z^{-1}\mathcal{L}_x^{-1}\mathcal{D}_y + \mathcal{L}_x^{-1}\mathcal{L}_y^{-1}\mathcal{D}_z.$$

so that

$$\left(-\frac{\nabla^2}{\alpha^2}\right)^m = \mathcal{C}_{xyz}^m \sum_{p=m}^{\infty} \binom{p-1}{m-1} \mathcal{D}_{xyz}^{p-m}.$$

[4] SINUM 2013, with A. Christlieb

$$
\begin{aligned}
u^{n+1} - 2u^n + u^{n-1} &= \sum_{m=1}^{\infty} \frac{2\beta^{2m}}{(2m)!} \left(\frac{\nabla^2}{\alpha^2}\right)^m u^n \\
&= \sum_{m=1}^{\infty} (-1)^m \frac{2\beta^{2m}}{(2m)!} \mathcal{C}^m \sum_{p=m}^{\infty} \binom{p-1}{m-1} \mathcal{D}^{p-m}[u^n] \\
&= \sum_{p=1}^{\infty} \sum_{m=1}^{p} (-1)^m \frac{2\beta^{2m}}{(2m)!} \binom{p-1}{m-1} \mathcal{C}^m \mathcal{D}^{p-m}[u^n] \\
&= \sum_{p=1}^{P} \sum_{m=1}^{p} A_{p,m}(\beta) \mathcal{C}^m \mathcal{D}^{p-m}[u^n] + O(\Delta t^{2P+2}).
\end{aligned}
$$

	$P = 1$			$P = 2$			$P = 3$		
Δt	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)
0.4	7.81E-1	*	1.5	7.19E-1	*	4.1	8.16E-1	*	8.2
0.2	2.46E-1	1.67	3.9	1.07E-1	2.74	11.2	7.80E-2	3.39	19.7
0.1	7.15E-2	1.78	7.1	1.03E-2	3.38	23.3	2.83E-3	4.78	44.1
0.05	1.89E-2	1.92	15.1	7.36E-4	3.81	48.3	5.74E-5	5.63	90.0
0.025	4.84E-3	1.96	30.0	4.97E-5	3.89	94.2	2.29E-6	4.64	186.2

Table: Refinement and computational efficiency for a 2d rectangular mode $u(x, y, 0) = \sin(\pi x)\sin(\pi y)$. The mesh is held fixed at $\Delta x = \Delta y = 0.003125$.

The algorithm scales linearly with the number of spatial points.

The characteristic polynomial satisfied by the amplification factor is

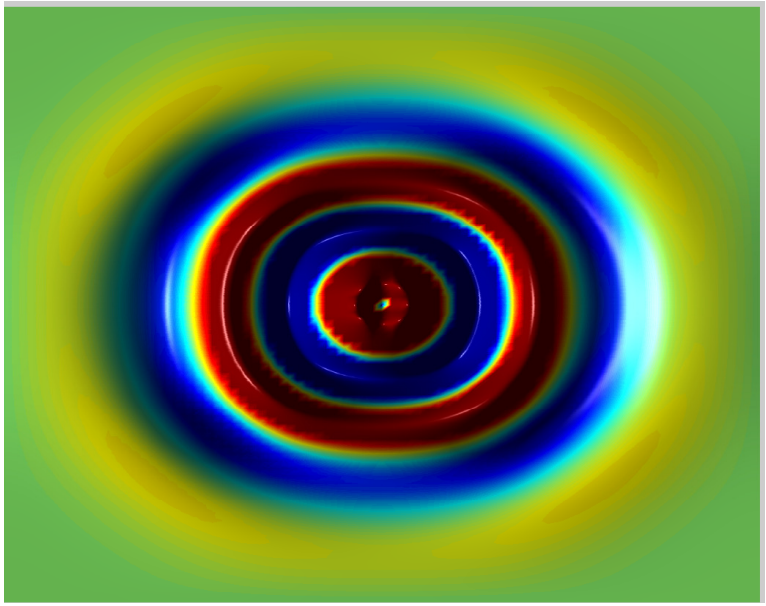
$$\rho^2 - 2\rho + 1 = \rho \left(\sum_{p=1}^{P} A_p(\beta) \hat{D}^p \right), \quad \beta > 0, \quad 0 \leq \hat{D} \leq 1.$$

Def. The scheme will be A-stable provided that $|\rho| \leq 1$.

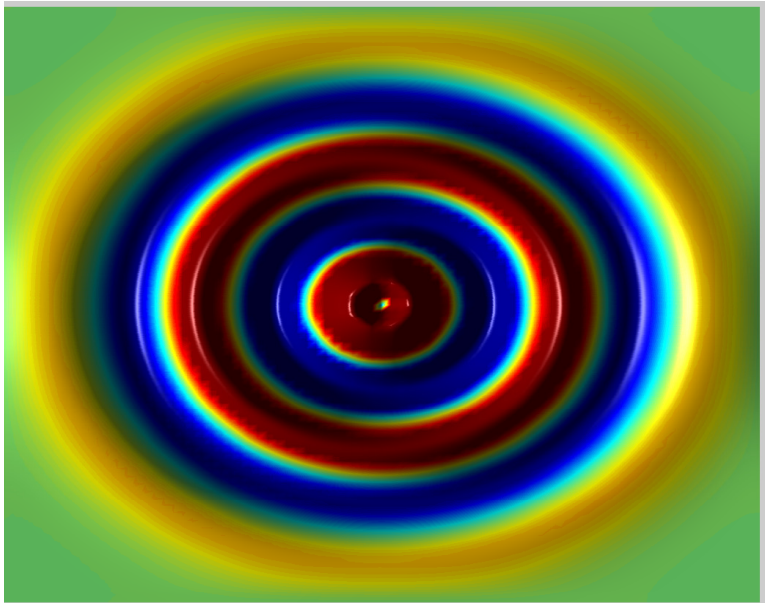
Theorem. For each finite P, there exists β_{max} such that the semi-discrete scheme will be A-stable for $0 < \beta \leq \beta_{max}$, and where

$$\sum_{p=1}^{P} A_p(\beta_{max}) = 4.$$

P	1	2	3	4	5
Order	2	4	6	8	10
β_{max}	2	1.4840	1.2345	1.0795	0.9715

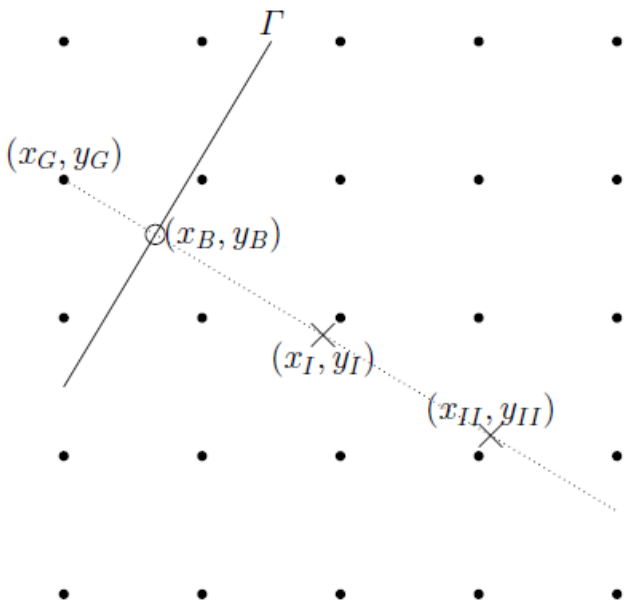


(a) 2nd order



(b) 4th order

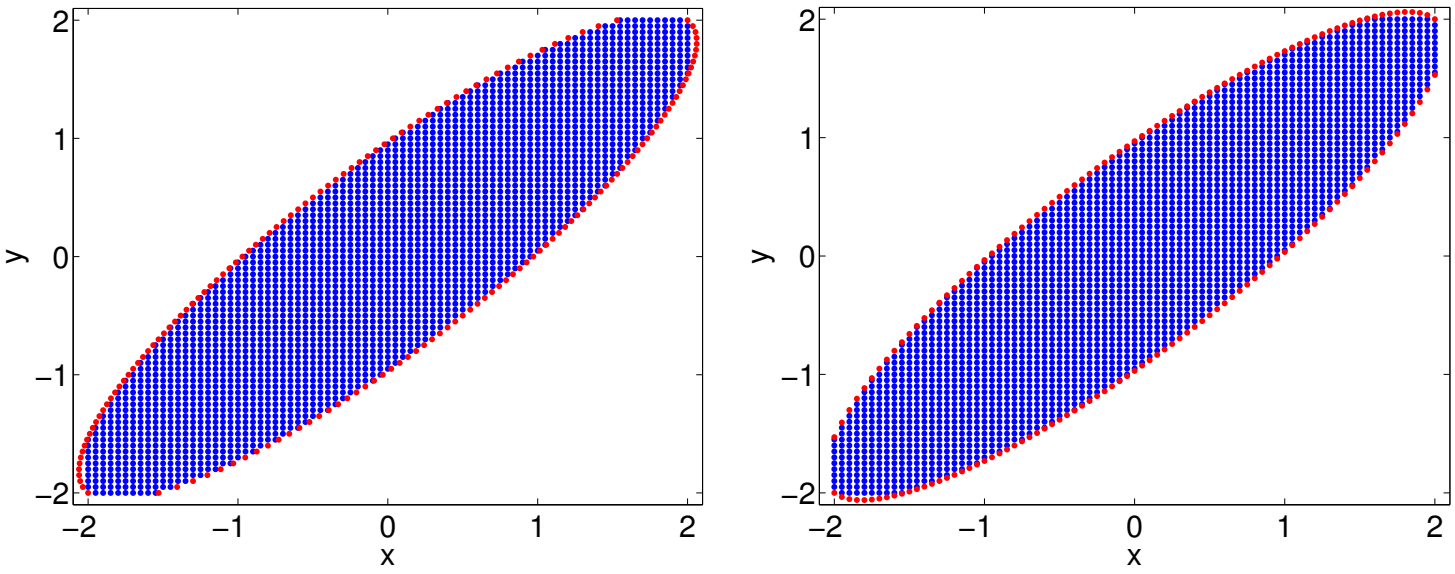
Figure: Propagation due to a point source in 2d, on a 80×80 mesh,with CFL number 2.



1. Add a **ghost point**, exterior to the domain.[5]
2. Interpolate u_I and u_{II} from **local** interior grid points (bilinear).
3. Interpolant through u_I and u_{II}, such that $\frac{\partial u}{\partial n}(\xi_B) = 0$.
4. Extrapolate to find u_G, and update **only** local grid points. Iterate to convergence.

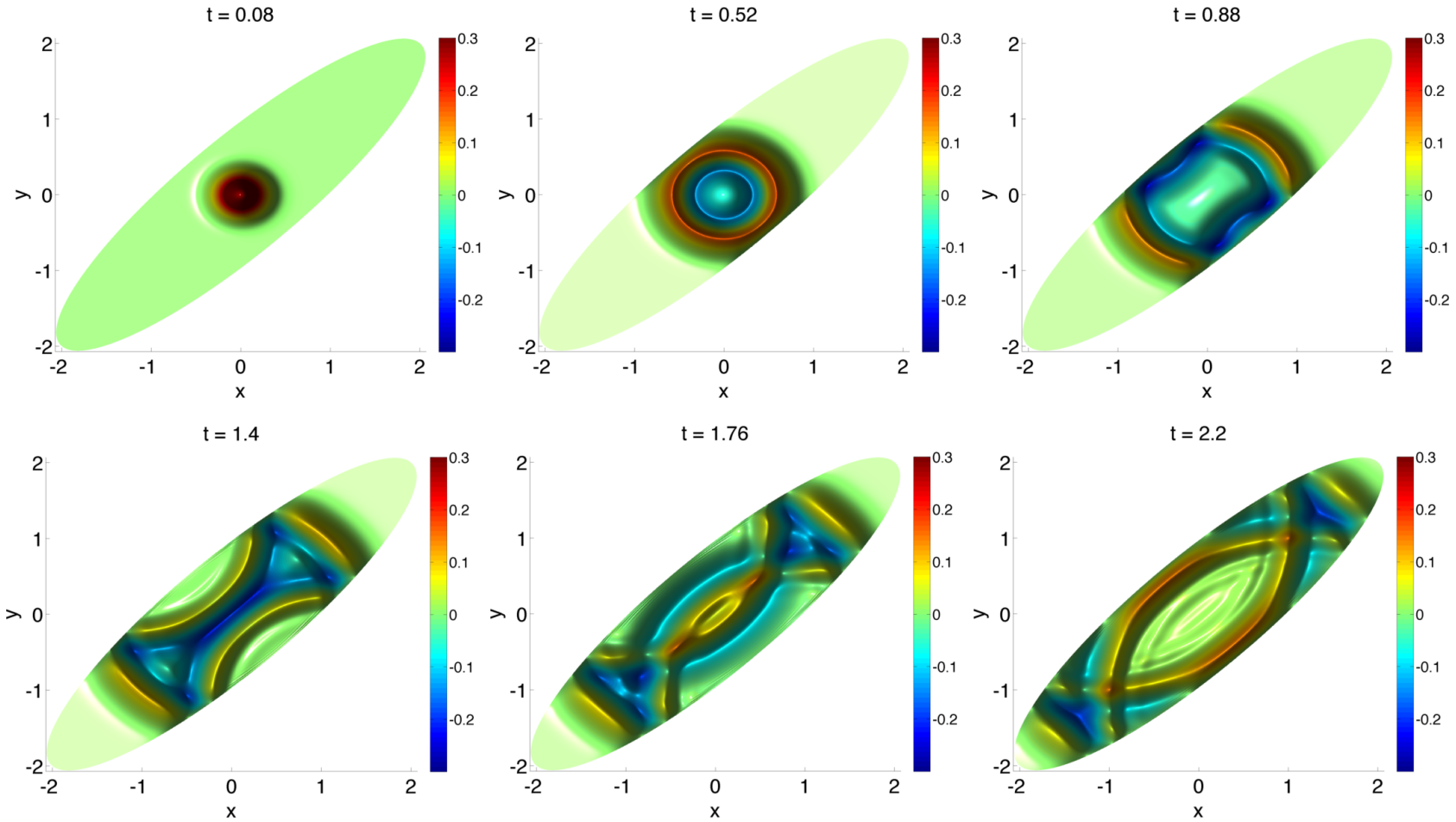
[5] J. Sci. Comp. 2017, with A. Christlieb and **E. Wolf**

$$C = \left\{ (x, y) : \left(\frac{x+y}{4} \right)^2 + (x-y)^2 = 1 \right\}.$$

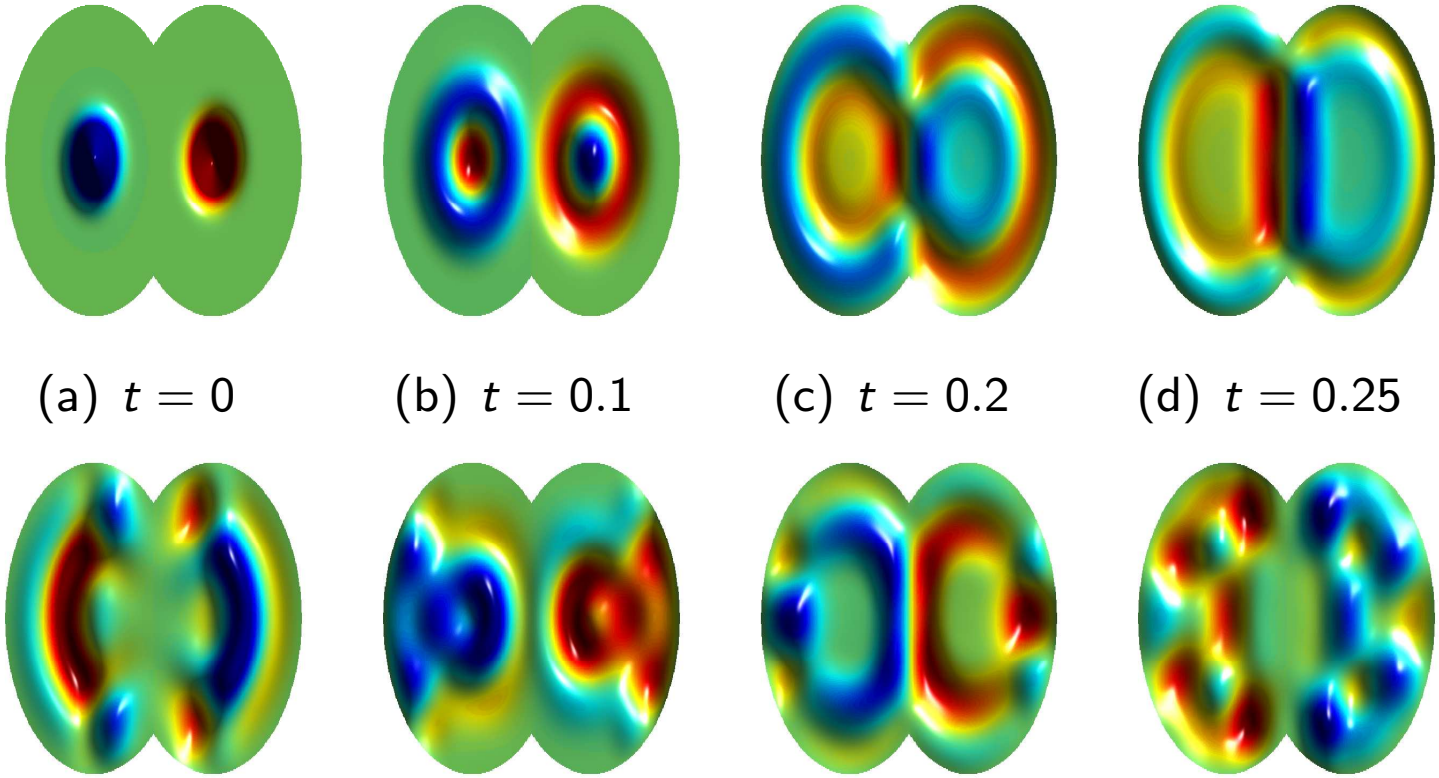


(a) x-sweep (b) y-sweep

Figure: The boundary points (red) close each line of the x and y sweeps.

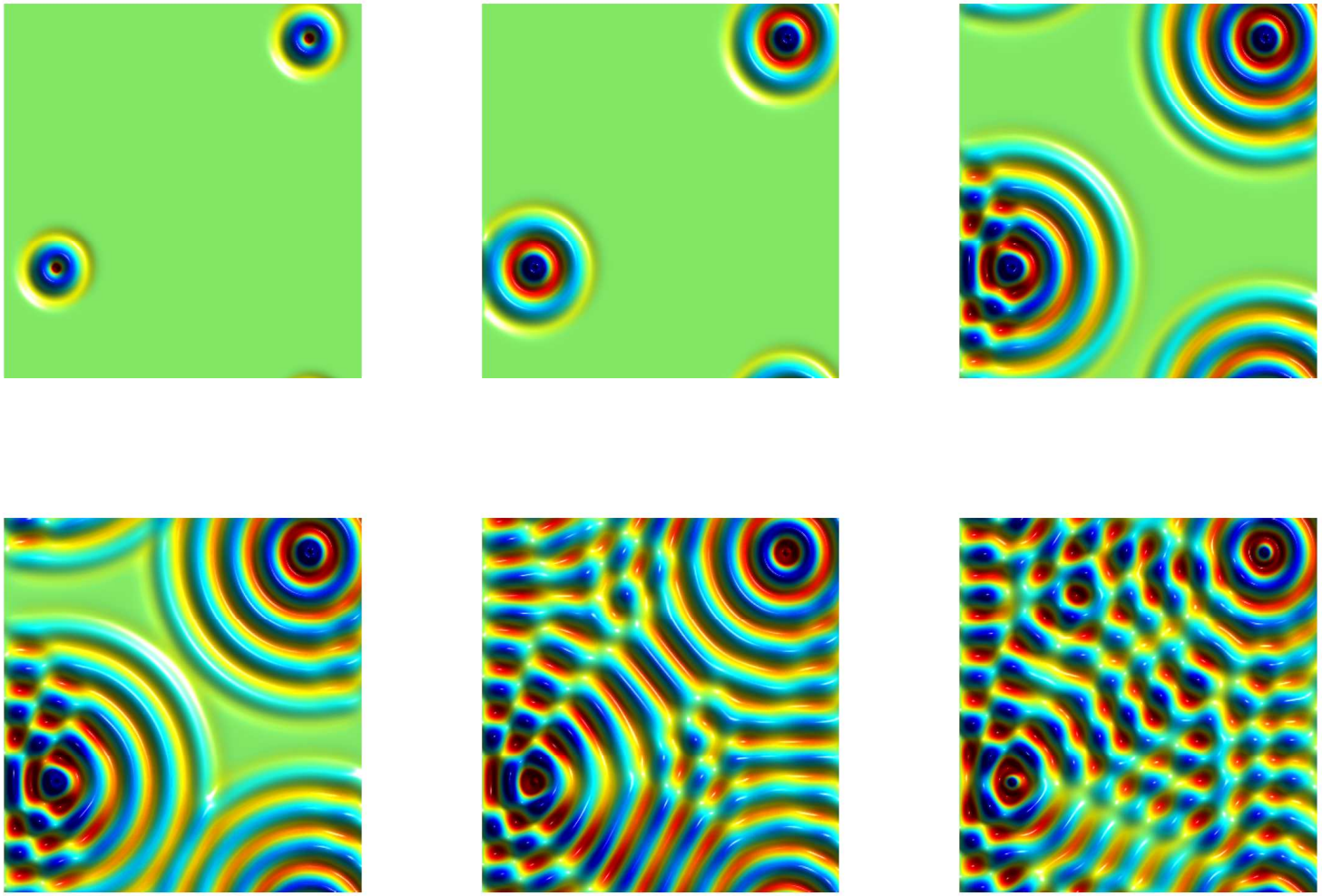


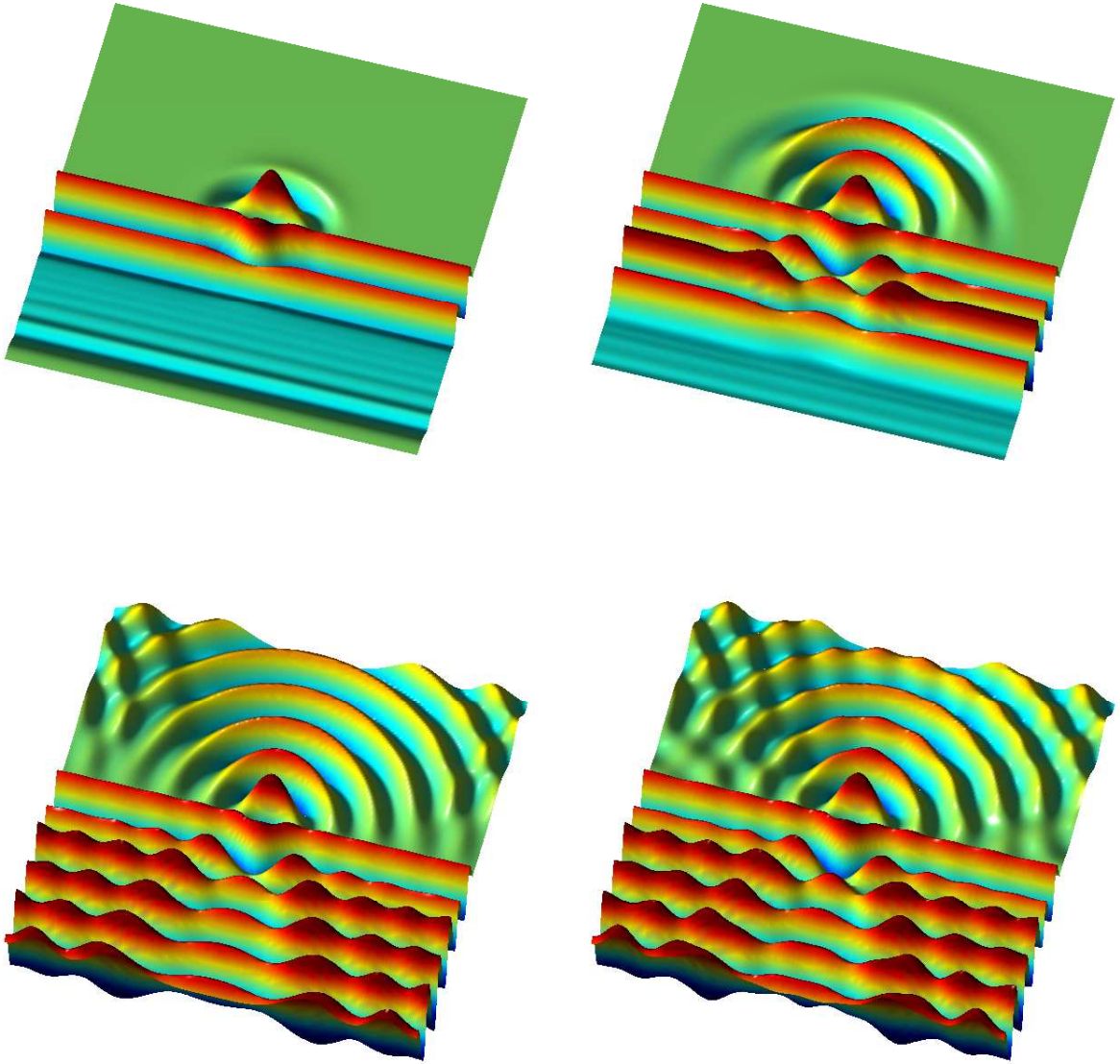
Solution is 6th order in time and space. No stability restriction.



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.25$

(e) $t = 0.45$ (f) $t = 0.7$ (g) $t = 0.8$ (h) $t = 1.0$





Implicit PIC, quasi-electrostatics.[6]

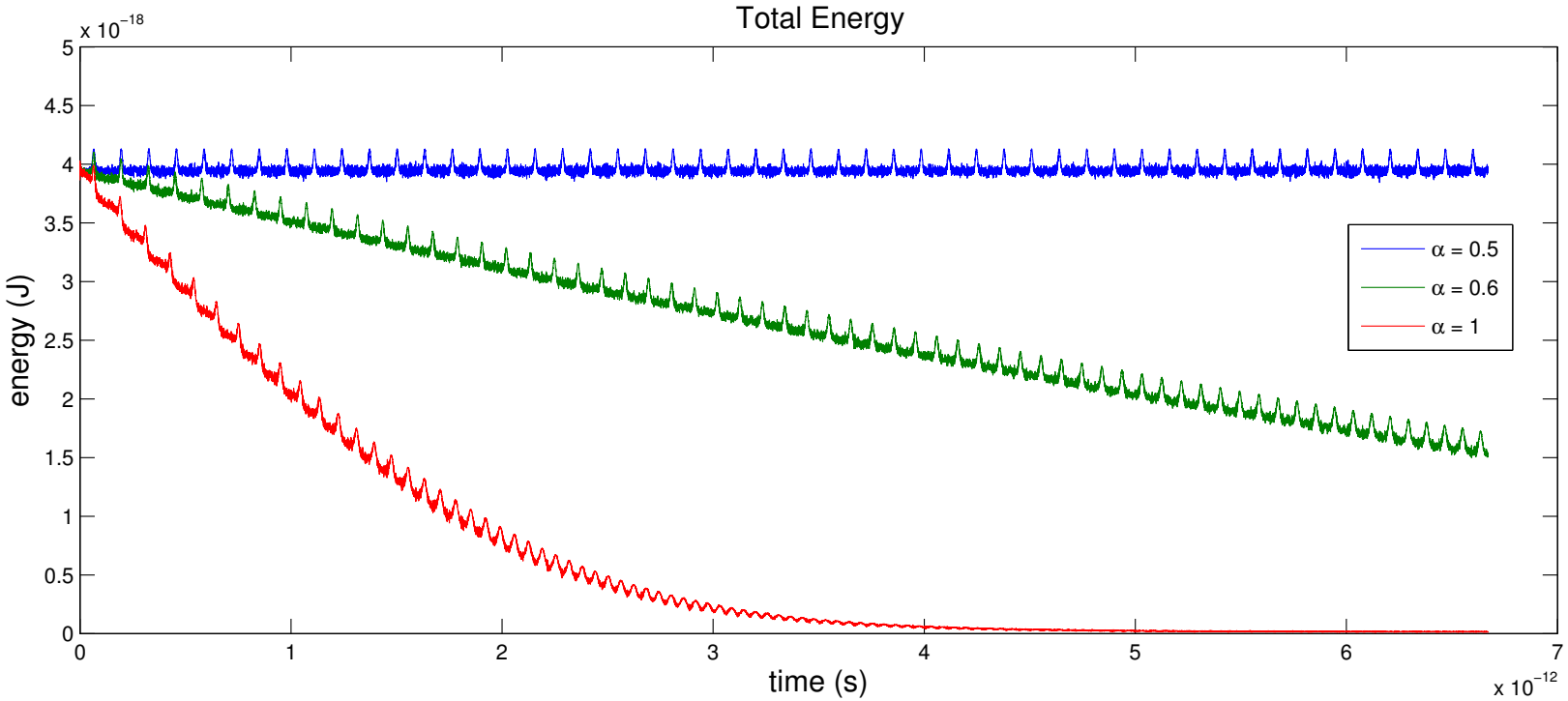
Initial Prediction

1. Charge: $\{\xi_i^n\} \to \rho^n$.
2. Potential (MOL^T): $\phi^{n-1}, \phi^n, \rho^n \to \phi^*$.
3. Positions: $\xi_i^* = \xi_i^n + \Delta t v_i^n$.
4. Fields: $\phi^* \to \vec{E}^* \to \vec{E}_i^* = \vec{E}^*(\xi_i^*)$.

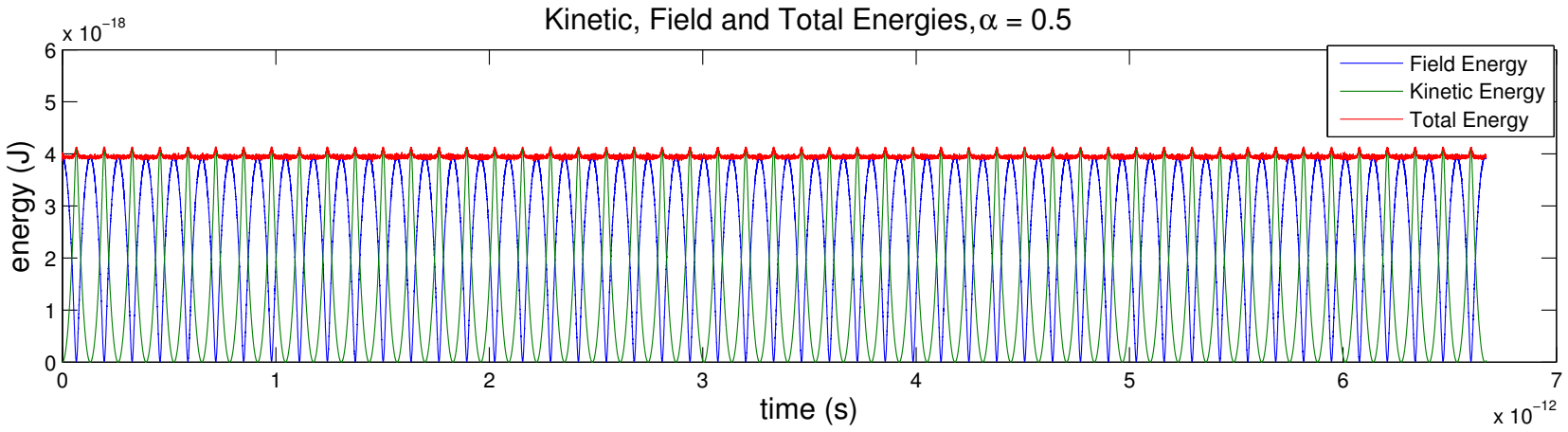
Correction Iteration

5. Velocities: $v_i^* = v_i^n + \Delta t(\alpha \vec{E}_i^* + (1-\alpha)\vec{E}_i^n)$.
6. Positions: $\xi_i^* = \xi_i^n + \Delta t(\alpha v_i^* + (1-\alpha)v_i^n)$.
7. Charge: $\{\xi_i^*\} \to \rho^*$.
8. Potential/Fields: $\phi^{n-1}, \phi^n, \bar{\rho} \to \phi^* \to \vec{E}_i^*$.
9. Repeat steps 5-8 to convergence.

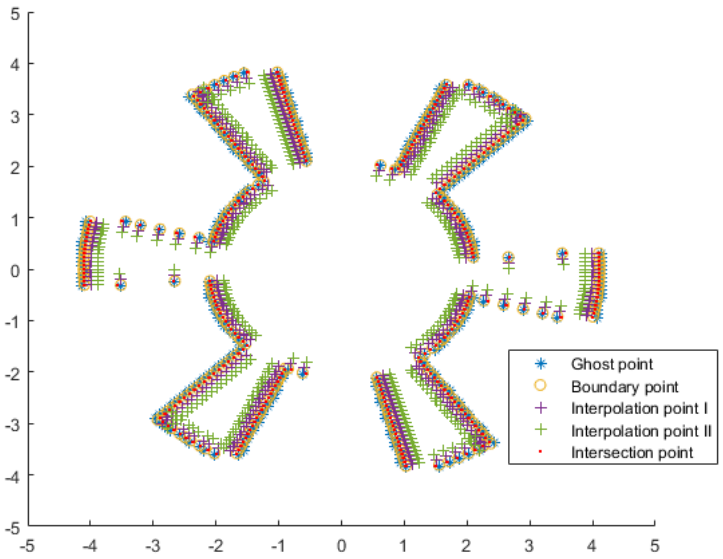
[6] JCP 2016, with M. Bettencourt A. Christlieb and **E. Wolf**

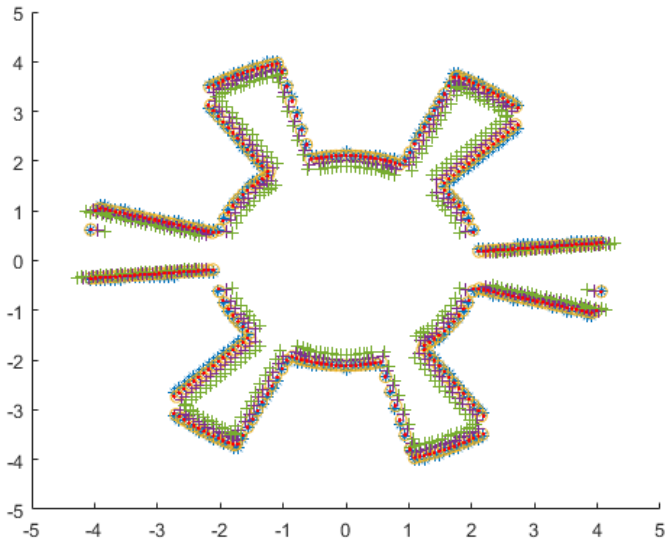


50 periods of an electron-ion oscillatory system.
The CFL is 10, with 100 cells in the domain. Standard spatial smoothing operators are applied to grid quantities.
The relaxation parameter is varied, no grid heating is observed when $\alpha = 0.5$.



Total energy is conserved, as the underlying field solver is non-dissipative.





Ph.D. thesis of M. Thavappiragasam

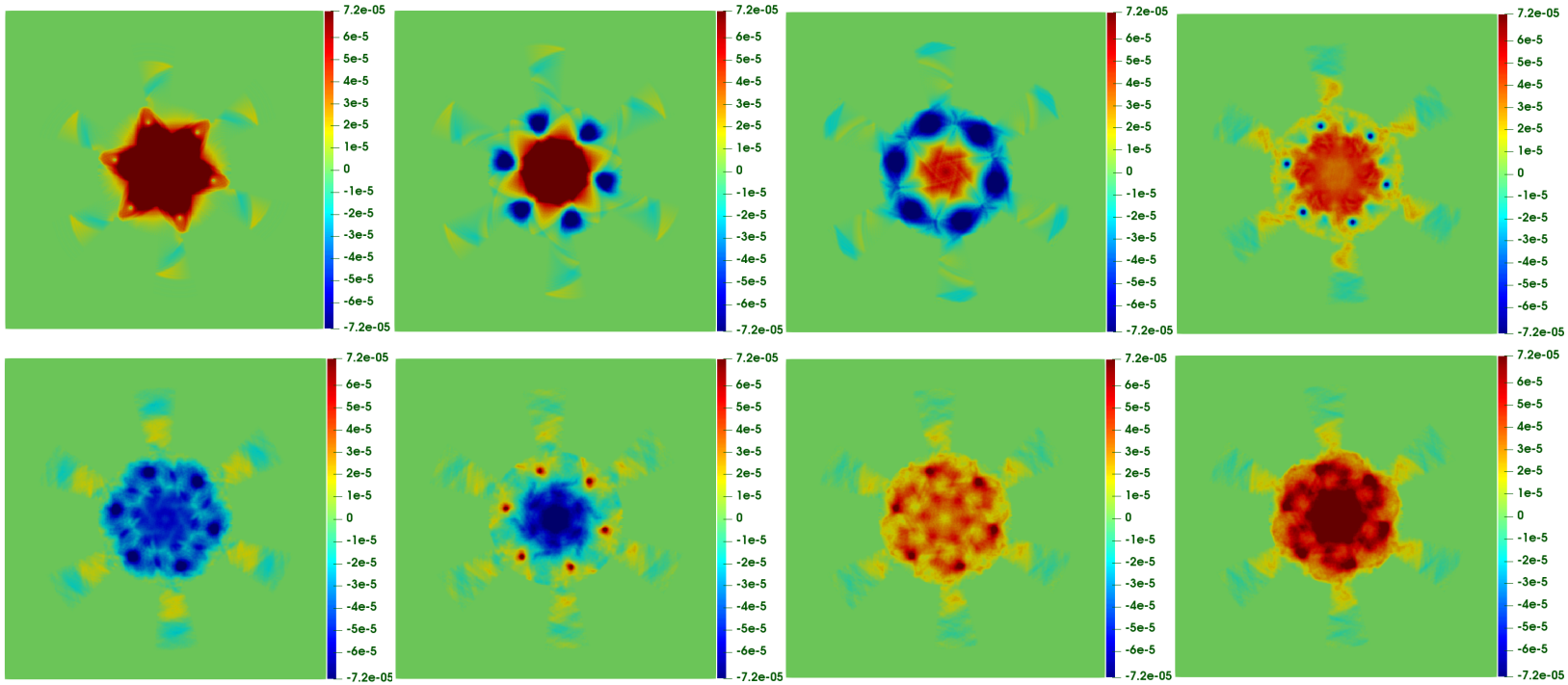


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Table of Contents

Future Work:

1. Analysis of outflow boundary conditions
2. Domain decomposition in multiple dimensions
3. Consistent treatment of particles
4. Fully parallel 3d Maxwell solver

Thank you!

References

The Christlieb Group